Topic 15:

Hypothesis test for Population Mean, μ, based on the sample mean, when σ is known

What do we know:

1) We have a population with mean μ and standard deviation σ.
2) We know the value of σ.
3) If we take repeated samples of size n and we look at the population of sample means, the $\bar{x}$'s, that population will have a normal distribution with mean μ and standard deviation $\sigma/\sqrt{n}$.
4) We have a null hypothesis, H_0, that claims to know the value of the population mean, μ.
5) We want to test that hypothesis against an alternative hypothesis, H_1, that we state.
6) We have set a level of significance, α, that is the probability of making a Type I error in doing that test.

Example: We know σ = 9.43. H_0: μ = 138 versus H_1: μ < 138. Take a sample of size 39 and use the sample mean, $\bar{x}$, to test this at the α = 0.05 level of significance.

Critical Value Approach

If H_0 is true, find the value y such that $P(X < y) = \alpha$ where X is the event of getting a sample mean.

For a N(0,1) the z value that has α in the left tail, i.e., less than z, is qnorm(α). qnorm(0.05)= -1.645

For a $N(\mu, \sigma/\sqrt{n})$ that value becomes $\mu + z^*\sigma/\sqrt{n}$ or in this case $138 + -1.645*9.43/\sqrt{39} = 135.516$

Therefore, 135.516 is our critical low value.

Now, take our sample. I turns out that the sample mean, $\bar{x}$, is 135.4. That value is less than our critical low value, 135.516. Therefore, we reject H_0 in favor of H_1.

Attained Significance Approach

Take our sample of size 39. Find the sample mean, $\bar{x}$. It turns out that $\bar{x}$=135.4.

Now ask, if H_0 is true, how strange is it to get a sample mean that low or more extreme (lower)?

We can find this via the command pnorm(135.4, mean=138, sd=9.43/sqrt(39)).

This produces 0.0425 which is less than α=0.05. Therefore, we reject H_0 in favor of H_1.

The steps that we take, in either approach, do not change from problem to problem. The only things that change are the values of H_0, σ, H_1, α, n, and $\bar{x}$. We might as well capture the steps in a function and just use the function to solve the problem.

```
> #   use the function to solve the problem
> source("../hypo_known.R")
> hypoth_test_known(138, 9.43, -1, 0.05, 39, 135.4)
              H0_mu                  H1:                sigma
              "138"          "mu <  138"               "9.43"
                  n            sig level            samp mean
               "39"               "0.05"              "135.4"
       sd samp mean             how far            test stat
 "1.51000849038197"   "2.48374294213 23"  "-1.7218446231003"
       critical low        critical high             attained
 "135.516257057868"               "n.a." "0.0425488343435557"
           decision
           "Reject"
```

Example: We know $\sigma = 3.94$. H_0: $\mu = 15.4$ versus H_1: $\mu > 15.4$. Take a sample of size 27 and use the sample mean, $\bar{x}$, to test this at the $\alpha = 0.02$ level of significance.

Critical Value Approach

This is another "one-sided" or "one-tailed" test. To "disprove" H_0 we need to get a sample with a mean that is too high to be believed. How high does it have to be?

For N(0,1) the z value that has 0.02 to its right is qnorm(0.02, lower.tail=FALSE) which gives 2.053749.

Use this to get the critical high value 15.4 + 2.053749 * 3.94/sqrt(27) which is 16.95726.

Now take the sample. Find the sample mean. It turns out to be 16.9, which is NOT greater than our critical high.

Therefore, we do not have enough evidence to reject H_0 in favor of H_1.

Attained Significance Approach

Take our sample. Find the sample mean. It turns out to be 16.9. Ask how strange would it be, if H_0 is true, for us to get s sample mean that high or higher?

pnorm(16.9, mean=15.4, sd=3.94/sqrt(27), lower.tail=FALSE)

This gives 0.02395135 which is NOT less than our level of significance.

Therefore, we do not have enough evidence to reject H_0 in favor of H_1.

Alternatively, we could use the function to do both approaches at once.

```
> #   or just use the function
> hypoth_test_known(15.4, 3.94, 1, 0.02, 27, 16.9)
             H0_mu                  H1:                sigma
            "15.4"          "mu >  15.4"               "3.94"
                 n             sig level            samp mean
              "27"               "0.02"               "16.9"
      sd samp mean              how far            test stat
"0.758253353535708"  "1.55726199880689"  "1.97823061778171"
      critical low         critical high             attained
            "n.a."  "16.9572619988069" "0.023951348079669"
          decision
    "do not reject"
```

Example: we know σ=17.4 we have H_0: μ=143.2 and H_1: $\mu \neq 143.2$ we will take a sample of size 42 do the test at $\alpha = 0.025$

This is a two-tailed test. By that we mean that we can reject H_0 if the sample mean is too high or too low. Therefore, in the case of the critical value approach we need to split the rejection area into halves, one in the lower tail and one in the upper tail. Also, in the attained significance approach if we find the probability of being so extreme in one direction then we need to double that to account for being so extreme in the opposite direction.

Critical Value Approach

This is a "two tailed" situation. In order to reject H_0 we would need to have the sample mean be too high or too low. How high or how low would we need these to be? Assuming H_0 is true, if yl is the low value and yh is the high value, then we want $P(\bar{x}<yl \text{ or } \bar{x}>yh)=0.025$. And, we want equal probability on each side, $P(\bar{x}<yl)=P(\bar{x}>yh)$.

For N(0,1) we find z such that $P(X<z)=0.025/2$ by qnorm(0.025/2). That value is -2.241403. By symmetry the z value such that $P(X>z)=0.025/2$ is 2.241403.

Taking these back to our $N(143.2,17.4/\sqrt{42})$ we get yl = $143.2 - 2.241403*(17.4/\sqrt{42})$ and we get yh $=143.2 + 2.241403*(17.4/\sqrt{42})$. yl is 137.1821 and yh is 149.2179.

Now we take our sample and we find that the mean of the sample, $\bar{x}$, is 149.3. That is higher than yh, our critical high value.

Therefore, we reject H_0 in favor of H_1 at the 0.025 level of significance.

Attained Significance Approach

We take our sample and we find that the sample mean, $\bar{x}$, is 149.3. Then, we ask, how strange would it be to get a mean this extreme or more extreme from a $N(143.2, 17/4/\sqrt{42})$ population.
We compute that via

```
pnorm( 149.3, mean=143.2,
    sd=17.4/sqrt(42), lower.tail=FALSE)
```

That turns out to be 0.01154374. But we are not done. This is a "two tailed" situation. We found the probability of being that high or higher, but we need to double that to account for being that extreme or lower on the other side. That means that our attained significance is 0.01154374*2 or 0.02308748.

That attained significance is lower than our level of signicance.

Therefore, we reject H_0 in favor of H_1 at the 0.025 level of significance.

Alternatively, we could use the function to do both approaches at once.

```
> hypoth_test_known( 143.2, 17.4, 0, 0.025, 42, 149.3 )
              H0_mu                  H1:                sigma
            "143.2"     "mu !=  143.2"               "17.4"
                  n            sig level            samp mean
               "42"              "0.025"              "149.3"
        sd samp mean              how far            test stat
  "2.6848782893404"  "6.01789352101487"  "2.27198380806253"
       critical low        critical high             attained
 "137.182106478985"  "149.217893521015" "0.0230874883249205"
           decision
           "Reject"
```

Example: We have a population with known $\sigma = 23.4$. We have H_0: $\mu = 178$ and H_1: $\mu > 178$. We want to run the test at the 0.05 level of significance. We take a sample of size 33. Here are the values in that sample.

gnrnd4(1895433204, 23401873)

204.5	162.7	166.7	141.5	150.3	170.6	211.2	178.2	231.2	164.8	142.9
229.9	150.5	162.1	168.0	188.9	200.6	179.8	189.3	218.7	179.0	209.5
198.7	164.7	203.1	180.5	212.0	189.0	185.1	174.4	166.0	217.1	195.4

To perform the test we need to find the mean, $\bar{x}$, of the values. We already know that the size of the sample is 33 and we have all the other values that we need to run the test.

```
> #   Example
> #     know sigma=23.4      H0: mean= 178     H1:  mean > 178
> #     perform the test at the 0.05 level of significance
> #   We take a sample of size 33.  The values in the sample
> #   can be generated using gnrnd4
> source("../gnrnd4.R")
> gnrnd4( 1895433204, 23401873)
style= 4   size= 33    seed= 89543   num digits= 1   alt_sign= 1
[1] "DONE "
> L1
 [1] 204.5 162.7 166.7 141.5 150.3 170.6 211.2 178.2 231.2 164.8 142.9 229.9 150.5 162.1
[15] 168.0 188.9 200.6 179.8 189.3 218.7 179.0 209.5 198.7 164.7 203.1 180.5 212.0 189.0
[29] 185.1 174.4 166.0 217.1 195.4
> x_bar <- mean(L1)
> x_bar
[1] 184.4515
> hypoth_test_known( 178, 23.4, 37, 0.05, 33, x_bar )
              H0_mu                    H1:                  sigma                     n
              "178"            "mu >  178"                 "23.4"                  "33"
          sig level              samp mean           sd samp mean               how far
             "0.05"    "184.451515151515"    "4.07341714936333"    "6.7001749722166"
          test stat           critical low          critical high              attained
 "1.58380910055419"                "n.a."   "184.700174972217" "0.0566185838495045"
           decision
     "do not reject"
```

Reading from the output of the function, we see that the critical high value is 184.700 and that the mean of the sample was 184.4515. Since $\bar{x}$ was not bigger than the the critical high value for this one-sided test we say that we do not have enough evidence to reject H_0 in favor of H_1 at the 0.05 level of significance.

Alternatively, we observe that the attained significance was 0.566. We were running the test at the 0.05 level of significance. 0.566 is not less than 0.05 so we say that we do not have enough evidence to reject H_0 in favor of H_1 at the 0.05 level of significance.